Heisenberg Model

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Abstract: In this research, we try to calculate analytically eigenvalues and eigenvectors of finite chain with 5 ½-spin particles Heisenberg model. We drove eigenfuctions for closed chain. After that we renormalized the result for interaction between chains for zero temperature. Finally according to self similarity, fix points were found from effective renormalization energy.

1. Calculation of Eigenenergy and Eigenstates of Finite Chain

1-1. 2 Length Chain

There are 4 states for 2 length open chain:

$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

Hamiltonian of this chain for open chain in XXY model is:

$$H = J\left(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \Delta\sigma_z^1\sigma_z^2\right)$$

Now we operate the Hamiltonian in the states:

	$\sigma_x^1\sigma_x^2$	$\sigma_y^1\sigma_y^2$	$\sigma_z^1\sigma_z^2$
$\|++\rangle$	$\|--\rangle$	$-\|--\rangle$	$\|++\rangle$
$\|+-\rangle$	$\|-+\rangle$	$\|-+\rangle$	$-\|+-\rangle$
$\|-+\rangle$	$\|+-\rangle$	$\|+-\rangle$	$-\|-+\rangle$
$\|--\rangle$	$\|++\rangle$	$-\|++\rangle$	$\|--\rangle$

so the operational form of Hamiltonian is

$$H = J\begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & -\Delta & 2 & 0 \\ 0 & 2 & -\Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix}$$

Eigenvalues and eigenstates are:

$\lambda = J\Delta \qquad :|++\rangle$

$\lambda = J(\Delta+2) \qquad :\frac{1}{2}\left(|+-\rangle + |-+\rangle\right)$

$\lambda = J(\Delta-2) \qquad :\frac{1}{2}\left(|+-\rangle - |-+\rangle\right)$

$\lambda = J\Delta \qquad :|--\rangle$

1-2. 3 Length Chain

There are 8 states:

$|+++\rangle, |++-\rangle, |+-+\rangle, |-++\rangle, |+--\rangle, |-+-\rangle, |--+\rangle, |---\rangle$

The Hamiltonian is:

$$H = J\left(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \Delta\sigma_z^1\sigma_z^2 + \sigma_x^2\sigma_x^3 + \sigma_y^2\sigma_y^3 + \Delta\sigma_z^2\sigma_z^3\right)$$

Because the chain is open the Hamiltonian operational matrix is block diagonal:

$$H = J\begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & H(m=\frac{1}{2}) & 0 & 0 \\ 0 & 0 & H(m=-\frac{1}{2}) & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix}$$

For $m = \frac{3}{2}$ and $m = -\frac{3}{2}$ eigenvalue are $J\Delta$ and obviously eigenfunctions are $|+++\rangle$ and $|---\rangle$. In other hand, for $m = \frac{1}{2}$ and $m = -\frac{1}{2}$ are symmetric and they are same. So it is enough to calculate eigenstates for $m = \frac{1}{2}$:

	$\sigma_x^1\sigma_x^2$	$\sigma_y^1\sigma_y^2$	$\sigma_z^1\sigma_z^2$	$\sigma_x^2\sigma_x^3$	$\sigma_y^2\sigma_y^3$	$\sigma_z^2\sigma_z^3$
$\|++-\rangle$	$\|---\rangle$	$-\|---\rangle$	$\|++-\rangle$	$\|+-+\rangle$	$\|+-+\rangle$	$-\|++-\rangle$
$\|+-+\rangle$	$\|-++\rangle$	$\|-++\rangle$	$-\|+-+\rangle$	$\|++-\rangle$	$\|++-\rangle$	$-\|+-+\rangle$
$\|-++\rangle$	$\|+-+\rangle$	$\|+-+\rangle$	$-\|-++\rangle$	$\|---\rangle$	$-\|---\rangle$	$\|-++\rangle$

So we write matrix operation:

$$H(m=\tfrac{1}{2}) = J\begin{pmatrix} 0 & 2 & 0 \\ 2 & -2\Delta & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

So eigenvalues and eigenstates are:

$\lambda = 0 \qquad : \frac{1}{2}\left(|++-\rangle - |-++\rangle\right)$

$\lambda = \left(-\Delta + \sqrt{\Delta^2+8}\right)J \qquad : \frac{4}{8+(\Delta-\sqrt{\Delta^2+8})^2 J^2}\left(|++-\rangle + |-++\rangle - \frac{1}{2}(\Delta-\sqrt{\Delta^2+8})J|+-+\rangle\right)$

$\lambda = \left(-\Delta - \sqrt{\Delta^2+8}\right)J \qquad : \frac{4}{8+(\Delta+\sqrt{\Delta^2+8})^2 J^2}\left(|++-\rangle + |-++\rangle - \frac{1}{2}(\Delta+\sqrt{\Delta^2+8})J|+-+\rangle\right)$

1-3. 5 Length Chain

The chain with 5 particles has 32 states and its Hamiltonian has block symmetry too. We want to find ground state, so it is enough to solving equation for $m = \frac{1}{2}$ and $m = -\frac{1}{2}$. Also these two have space symmetry, so we solve the equations for $m = \frac{1}{2}$ then we extend the result to $m = -\frac{1}{2}$. The states which we have to solve are:

$|+++--\rangle, |++-+-\rangle, |++--+\rangle, |+-++-\rangle, |+-+-+\rangle$

$|--+++\rangle, |-+-++\rangle, |+--++\rangle, |-++-+\rangle, |-+++-\rangle$

1-3-1. Open Chain

Hamiltonian for open chain is:

$$H = J\left(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \Delta\sigma_z^1\sigma_z^2 + \sigma_x^2\sigma_x^3 + \sigma_y^2\sigma_y^3 + \Delta\sigma_z^2\sigma_z^3 + \sigma_x^3\sigma_x^4 + \sigma_y^3\sigma_y^4 + \Delta\sigma_z^3\sigma_z^4 + \sigma_x^4\sigma_x^5 + \sigma_y^4\sigma_y^5 + \Delta\sigma_z^4\sigma_z^5\right)$$

	$\sigma_x^1\sigma_x^2$	$\sigma_y^1\sigma_y^2$	$\sigma_z^1\sigma_z^2$	$\sigma_x^2\sigma_x^3$	$\sigma_y^2\sigma_y^3$	$\sigma_z^2\sigma_z^3$	$\sigma_x^3\sigma_x^4$	$\sigma_y^3\sigma_y^4$	$\sigma_z^3\sigma_z^4$	$\sigma_x^4\sigma_x^5$	$\sigma_y^4\sigma_y^5$	$\sigma_z^4\sigma_z^5$
$\vert +++--\rangle$	$\vert --+--\rangle$	$-\vert --+--\rangle$	$\vert +++--\rangle$	$\vert +----\rangle$	$-\vert +----\rangle$	$\vert +++--\rangle$	$\vert ++-+-\rangle$	$\vert ++-+-\rangle$	$-\vert +++--\rangle$	$\vert +++++\rangle$	$-\vert +++++\rangle$	$\vert +++--\rangle$
$\vert ++-+-\rangle$	$\vert ---+-\rangle$	$-\vert ---+-\rangle$	$\vert ++-+-\rangle$	$\vert +-++-\rangle$	$\vert +-++-\rangle$	$-\vert ++-+-\rangle$	$\vert +++--\rangle$	$\vert +++--\rangle$	$-\vert ++-+-\rangle$	$\vert ++--+\rangle$	$\vert ++--+\rangle$	$-\vert ++-+-\rangle$
$\vert ++--+\rangle$	$\vert ----+\rangle$	$-\vert ----+\rangle$	$\vert ++--+\rangle$	$\vert +-+-+\rangle$	$\vert +-+-+\rangle$	$-\vert ++--+\rangle$	$\vert +++++\rangle$	$-\vert +++++\rangle$	$\vert ++--+\rangle$	$\vert ++-+-\rangle$	$\vert ++-+-\rangle$	$-\vert ++--+\rangle$
$\vert +-++-\rangle$	$\vert -+++-\rangle$	$\vert -+++-\rangle$	$-\vert +-++-\rangle$	$\vert ++-+-\rangle$	$\vert ++-+-\rangle$	$-\vert +-++-\rangle$	$\vert +----\rangle$	$-\vert +----\rangle$	$\vert +-++-\rangle$	$\vert +-+-+\rangle$	$\vert +-+-+\rangle$	$-\vert +-++-\rangle$
$\vert +-+-+\rangle$	$\vert -++-+\rangle$	$\vert -++-+\rangle$	$-\vert +-+-+\rangle$	$\vert ++--+\rangle$	$\vert ++--+\rangle$	$-\vert +-+-+\rangle$	$\vert +--++\rangle$	$\vert +--++\rangle$	$-\vert +-+-+\rangle$	$\vert +-++-\rangle$	$\vert +-++-\rangle$	$-\vert +-+-+\rangle$
$\vert -+++-\rangle$	$\vert +-++-\rangle$	$\vert +-++-\rangle$	$-\vert -+++-\rangle$	$\vert ---+-\rangle$	$-\vert ---+-\rangle$	$\vert -+++-\rangle$	$\vert -+---\rangle$	$-\vert -+---\rangle$	$\vert -+++-\rangle$	$\vert -++-+\rangle$	$\vert -++-+\rangle$	$\vert -+++-\rangle$
$\vert -++-+\rangle$	$\vert +-+-+\rangle$	$\vert +-+-+\rangle$	$\vert -++-+\rangle$	$\vert ----+\rangle$	$-\vert ----+\rangle$	$\vert -++-+\rangle$	$\vert -+-++\rangle$	$\vert -+-++\rangle$	$-\vert -++-+\rangle$	$\vert -+++-\rangle$	$\vert -+++-\rangle$	$-\vert -++-+\rangle$
$\vert +--++\rangle$	$\vert -+-++\rangle$	$\vert -+-++\rangle$	$-\vert +--++\rangle$	$\vert +++++\rangle$	$-\vert +++++\rangle$	$\vert +--++\rangle$	$\vert +-+-+\rangle$	$\vert +-+-+\rangle$	$-\vert +--++\rangle$	$\vert +----\rangle$	$-\vert +----\rangle$	$\vert +--++\rangle$
$\vert -+-++\rangle$	$\vert +--++\rangle$	$\vert +--++\rangle$	$-\vert -+-++\rangle$	$\vert --+++\rangle$	$\vert --+++\rangle$	$-\vert -+-++\rangle$	$\vert -++-+\rangle$	$\vert -++-+\rangle$	$-\vert -+-++\rangle$	$\vert -+---\rangle$	$-\vert -+---\rangle$	$\vert -+-++\rangle$
$\vert --+++\rangle$	$\vert +++++\rangle$	$-\vert +++++\rangle$	$\vert --+++\rangle$	$\vert -+-++\rangle$	$\vert -+-++\rangle$	$-\vert --+++\rangle$	$\vert ----+\rangle$	$-\vert ----+\rangle$	$\vert --+++\rangle$	$\vert --+--\rangle$	$-\vert --+--\rangle$	$\vert --+++\rangle$

So Hamiltonian matrix is:

$$H=2J\begin{pmatrix} \Delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\Delta & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\Delta & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2\Delta & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -\Delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -\Delta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \Delta \end{pmatrix}=2Jh$$

Calculation of Eigenvalue:
The Hamiltonian equation is

$$H\psi = \lambda\psi = 2Jh\psi = 2J\lambda'\psi \Rightarrow (h-\lambda'1)\psi = 0 \Rightarrow \det\left(h-\lambda'1\right) = 0$$

The determinant is expanded to 10 degree equation:

$$-4\Delta^2 - 8\Delta^4 + 12\Delta^3\lambda + 16\Delta^5\lambda + 12\lambda^2 + 40\Delta^2\lambda^2 + 7\Delta^4\lambda^2 - 10\Delta^6\lambda^2 - 28\Delta\lambda^3 - 64\Delta^3\lambda^3 - 18\Delta^5\lambda^3 + 2\Delta^7\lambda^3 - 40\lambda^4 - 42\Delta^2\lambda^4 + 20\Delta^4\lambda^4 + 5\Delta^6\lambda^4 + 56\Delta\lambda^5 + 48\Delta^3\lambda^5 + 39\lambda^6 + 2\Delta^2\lambda^6 - 9\Delta^4\lambda^6 - 30\Delta\lambda^7 - 6\Delta^3\lambda^7 - 12\lambda^8 + 3\Delta^2\lambda^8 + 4\Delta\lambda^9 + \lambda^{10} == 0$$

To solving this equation, we have to use vector form to finding eigenvectors:

$$\begin{pmatrix} \Delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\Delta & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\Delta & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2\Delta & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -\Delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -\Delta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \Delta \end{pmatrix}\psi = \lambda\psi \text{ and } \psi = \begin{pmatrix} a1 \\ a2 \\ a3 \\ a4 \\ a5 \\ a6 \\ a7 \\ a8 \\ a9 \\ a10 \end{pmatrix}$$

Now we use the symmetry to determine eigenvector's elements in term of each other.
$a_1 \equiv \alpha$
$a_2 + a_1\Delta = \lambda a_1 \rightarrow a_2 = (\lambda-\Delta)\alpha$
$a_1 - \Delta a_2 + a_3 + a_4 = \lambda a_2 \rightarrow a_3 + a_4 = a_1\left((\lambda-\Delta)\lambda + (\lambda-\Delta)\Delta - 1\right) = \left((\lambda-\Delta)(\lambda+\Delta)-1\right)\alpha$
$\Delta a_4 - a_6 = \lambda(a_3 - a_4)$

$$a_6 \equiv \beta \rightarrow \begin{cases} a_3 + a_4 = ((\lambda - \Delta)(\lambda + \Delta) - 1)\alpha \\ \lambda(a_3 - a_4) = \Delta a_4 - \beta \end{cases} \rightarrow \begin{cases} a_3 = \dfrac{\lambda(\lambda+\Delta)\left[(\lambda-\Delta)(\lambda+\Delta)-1\right]}{2\lambda+\Delta}\alpha - \dfrac{\beta}{2\lambda+\Delta} \\ a_4 = \dfrac{\lambda\left[(\lambda-\Delta)(\lambda+\Delta)-1\right]}{2\lambda+\Delta}\alpha + \dfrac{\beta}{2\lambda+\Delta} \end{cases}$$

$$a_{10} \equiv \gamma \rightarrow a_9 = (\lambda - \Delta)\gamma \rightarrow \begin{cases} a_8 = \dfrac{\lambda(\lambda+\Delta)\left[(\lambda-\Delta)(\lambda+\Delta)-1\right]}{2\lambda+\Delta}\gamma - \dfrac{\beta}{2\lambda+\Delta} \\ a_7 = \dfrac{\lambda\left[(\lambda-\Delta)(\lambda+\Delta)-1\right]}{2\lambda+\Delta}\gamma + \dfrac{\beta}{2\lambda+\Delta} \end{cases}$$

To evaluate 5th element, the eigenvalue equations for 3rd and 8th elements are:

$$\begin{cases} \lambda a_3 = a_2 + a_5 \\ \lambda a_8 = a_9 + a_5 \end{cases} \Rightarrow \begin{cases} a_5 = \lambda a_3 - a_2 = \dfrac{\lambda(\lambda+\Delta)\left[(\lambda+\Delta)(\lambda-\Delta)-1\right]}{2\lambda+\Delta}\alpha - \dfrac{\lambda}{2\lambda+\Delta}\beta - (\lambda-\Delta)\alpha \\ a_5 = \lambda a_8 - a_9 = \dfrac{\lambda(\lambda+\Delta)\left[(\lambda+\Delta)(\lambda-\Delta)-1\right]}{2\lambda+\Delta}\gamma - \dfrac{\lambda}{2\lambda+\Delta}\beta - (\lambda-\Delta)\gamma \end{cases}$$

So,

$$\left\{\frac{\lambda(\lambda+\Delta)\left[(\lambda+\Delta)(\lambda-\Delta)-1\right]}{2\lambda+\Delta} - (\lambda-\Delta)\right\}(\alpha-\gamma) = 0$$

means the solutions have been satisfy these conditions:

$$\left\{\frac{\lambda(\lambda+\Delta)\left[(\lambda+\Delta)(\lambda-\Delta)-1\right]}{2\lambda+\Delta} - (\lambda-\Delta)\right\} = 0 \text{ or } \alpha = \gamma$$

Four eigenvalues are arrived from $\left\{\dfrac{\lambda(\lambda+\Delta)\left[(\lambda+\Delta)(\lambda-\Delta)-1\right]}{2\lambda+\Delta} - (\lambda-\Delta)\right\} = 0$:

$$\lambda \rightarrow -\frac{\Delta}{4} - \frac{1}{2}\sqrt{\left(3 + \frac{5\Delta^2}{4} + \frac{1}{3}(-3-\Delta^2) + \frac{9+18\Delta^2+4\Delta^4}{3\left(-27+81\Delta^2+27\Delta^4+8\Delta^6-3\sqrt{3}\sqrt{-324\Delta^2-171\Delta^4-214\Delta^6-85\Delta^8-16\Delta^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\Delta^2+27\Delta^4+8\Delta^6-3\sqrt{3}\sqrt{-324\Delta^2-171\Delta^4-214\Delta^6-85\Delta^8-16\Delta^{10}}\right)^{1/3}\right)} -$$

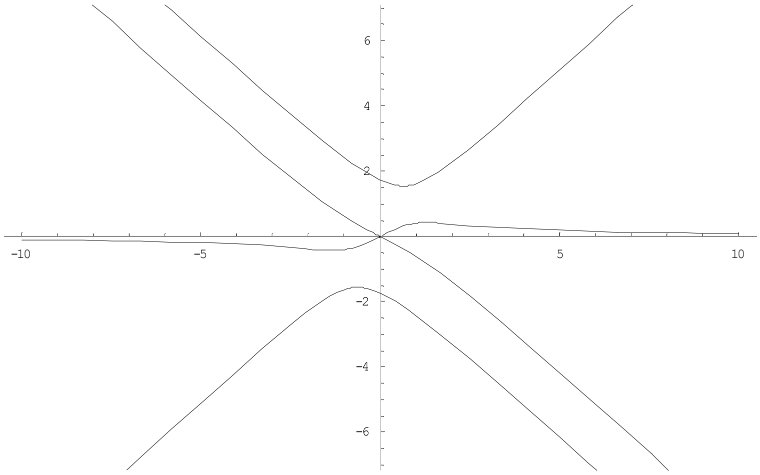
$$\frac{1}{2}\sqrt{\left(3 + \frac{3\Delta^2}{2} + \frac{1}{3}(3+\Delta^2) - \frac{9+18\Delta^2+4\Delta^4}{3\left(-27+81\Delta^2+27\Delta^4+8\Delta^6-3\sqrt{3}\sqrt{-324\Delta^2-171\Delta^4-214\Delta^6-85\Delta^8-16\Delta^{10}}\right)^{1/3}} - \frac{1}{3}\left(-27+81\Delta^2+27\Delta^4+8\Delta^6-3\sqrt{3}\sqrt{-324\Delta^2-171\Delta^4-214\Delta^6-85\Delta^8-16\Delta^{10}}\right)^{1/3} - \left(7\Delta^3+4\Delta(-3-\Delta^2)\right) \middle/ \left(4\sqrt{\left(3 + \frac{5\Delta^2}{4} + \frac{1}{3}(-3-\Delta^2) + \frac{9+18\Delta^2+4\Delta^4}{3\left(-27+81\Delta^2+27\Delta^4+8\Delta^6-3\sqrt{3}\sqrt{-324\Delta^2-171\Delta^4-214\Delta^6-85\Delta^8-16\Delta^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\Delta^2+27\Delta^4+8\Delta^6-3\sqrt{3}\sqrt{-324\Delta^2-171\Delta^4-214\Delta^6-85\Delta^8-16\Delta^{10}}\right)^{1/3}\right)}\right)\right)}$$

$$\lambda \to -\frac{\triangle}{4} - \frac{1}{2}\sqrt{\left(3 + \frac{5\triangle^2}{4} + \frac{1}{3}\left(-3-\triangle^2\right) + \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}\right)} + \frac{1}{2}\sqrt{\left(3 + \frac{3\triangle^2}{2} + \frac{1}{3}\left(3+\triangle^2\right) - \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} - \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3} - \left(7\triangle^3+4\triangle\left(-3-\triangle^2\right)\right) \Big/ \left(4\sqrt{\left(3 + \frac{5\triangle^2}{4} + \frac{1}{3}\left(-3-\triangle^2\right) + \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}\right)}\right)\right)}$$

$$\lambda \to -\frac{\triangle}{4} + \frac{1}{2}\sqrt{\left(3 + \frac{5\triangle^2}{4} + \frac{1}{3}\left(-3-\triangle^2\right) + \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}\right)} - \frac{1}{2}\sqrt{\left(3 + \frac{3\triangle^2}{2} + \frac{1}{3}\left(3+\triangle^2\right) - \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} - \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3} + \left(7\triangle^3+4\triangle\left(-3-\triangle^2\right)\right) \Big/ \left(4\sqrt{\left(3 + \frac{5\triangle^2}{4} + \frac{1}{3}\left(-3-\triangle^2\right) + \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}\right)}\right)\right)}$$

$$\lambda \to -\frac{\triangle}{4} + \frac{1}{2}\sqrt{\left(3 + \frac{5\triangle^2}{4} + \frac{1}{3}\left(-3-\triangle^2\right) + \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}\right)} + \frac{1}{2}\sqrt{\left(3 + \frac{3\triangle^2}{2} + \frac{1}{3}\left(3+\triangle^2\right) - \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} - \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3} + \left(7\triangle^3+4\triangle\left(-3-\triangle^2\right)\right) \Big/ \left(4\sqrt{\left(3 + \frac{5\triangle^2}{4} + \frac{1}{3}\left(-3-\triangle^2\right) + \frac{9+18\triangle^2+4\triangle^4}{3\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}} + \frac{1}{3}\left(-27+81\triangle^2+27\triangle^4+8\triangle^6-3\sqrt{3}\sqrt{-324\triangle^2-171\triangle^4-214\triangle^6-85\triangle^8-16\triangle^{10}}\right)^{1/3}\right)}\right)\right)}$$

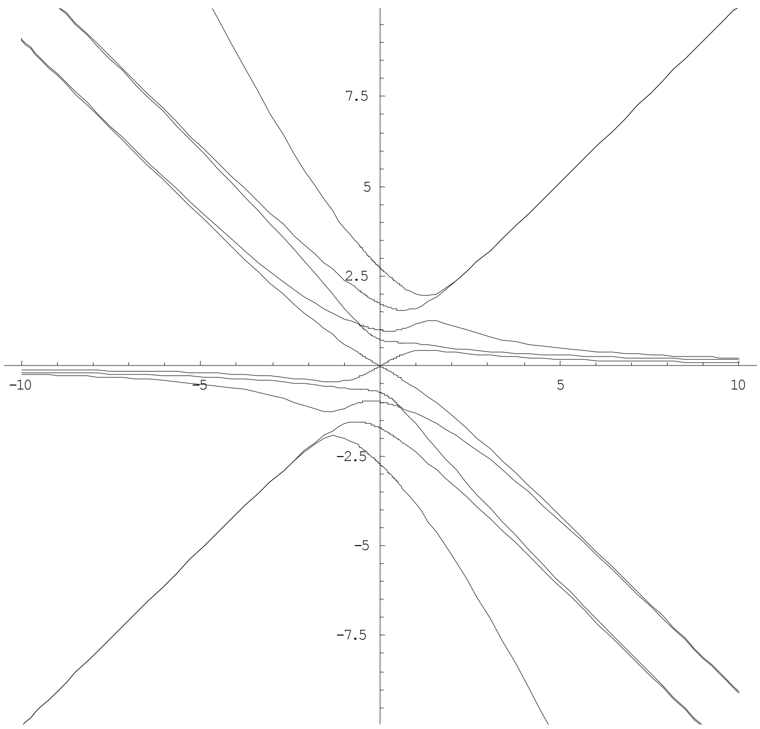
The graph of these four values is:



But for $\alpha = \gamma$, there are six solution which will be arrived from a six degree education:

$$-4 - 8\Delta^2 + 8\Delta\lambda + 8\Delta^3\lambda + 12\lambda^2 + 7\Delta^2\lambda^2 - 2\Delta^4\lambda^2 - 12\Delta\lambda^3 - 3\Delta^3\lambda^3 - 9\lambda^4 + \Delta^2\lambda^4 + 3\Delta\lambda^5 + \lambda^6 = 0$$

which should be solved by Mathematica. The graph of all eigenvalues in term of Δ are scratched in this graph:



1-3-2. Closed Chain

For chain, we rewrite the Hamiltonian:

$$H = J\big(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \Delta\sigma_z^1\sigma_z^2 + \sigma_x^2\sigma_x^3 + \sigma_y^2\sigma_y^3 + \Delta\sigma_z^2\sigma_z^3 + \sigma_x^3\sigma_x^4 + \sigma_y^3\sigma_y^4 + \Delta\sigma_z^3\sigma_z^4$$
$$+ \sigma_x^4\sigma_x^5 + \sigma_y^4\sigma_y^5 + \Delta\sigma_z^4\sigma_z^5 + \sigma_x^5\sigma_x^1 + \sigma_y^5\sigma_y^1 + \Delta\sigma_z^5\sigma_z^1\big)$$

	$\sigma_x^5\sigma_x^1$	$\sigma_y^5\sigma_y^1$	$\sigma_z^5\sigma_z^1$
$\lvert +++--\rangle$	$\lvert -++-+\rangle$	$\lvert -++-+\rangle$	$-\lvert +++--\rangle$
$\lvert ++-+-\rangle$	$\lvert -+-++\rangle$	$\lvert -+-++\rangle$	$-\lvert ++-+-\rangle$
$\lvert ++--+\rangle$	$\lvert -+---\rangle$	$-\lvert -+---\rangle$	$\lvert ++--+\rangle$
$\lvert +-++-\rangle$	$\lvert --+++\rangle$	$\lvert --+++\rangle$	$-\lvert +-++-\rangle$
$\lvert +-+-+\rangle$	$\lvert --+--\rangle$	$-\lvert --+--\rangle$	$\lvert +-+-+\rangle$
$\lvert -+++-\rangle$	$\lvert +++++\rangle$	$-\lvert +++++\rangle$	$\lvert -+++-\rangle$
$\lvert -++-+\rangle$	$\lvert +++--\rangle$	$\lvert +++--\rangle$	$-\lvert -++-+\rangle$
$\lvert +--++\rangle$	$\lvert ---+-\rangle$	$-\lvert ---+-\rangle$	$\lvert +--++\rangle$
$\lvert -+-++\rangle$	$\lvert ++-+-\rangle$	$\lvert ++-+-\rangle$	$-\lvert -+-++\rangle$
$\lvert --+++\rangle$	$\lvert +-++-\rangle$	$\lvert +-++-\rangle$	$-\lvert --+++\rangle$

and Hamiltonian operational matrix become:

$$H=J\begin{pmatrix}
\Delta & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
2 & -3\Delta & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 2 & \Delta & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -3\Delta & 2 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & 2 & 2 & -3\Delta & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & \Delta & 2 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 2 & 2 & -3\Delta & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & \Delta & 2 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & -3\Delta & 2 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & \Delta
\end{pmatrix}$$

Calculate Eigenvalue:
In closed chain, the periodic condition makes a rotational symmetry. We call T the rotation operation. So all of states of will be categorize in two group. The states are shown in new form:

$\left\|-+++-\right\rangle$	$\left\|+-+-+\right\rangle$
$\left\|--+++\right\rangle = T\left\|-+++-\right\rangle$	$\left\|++-+-\right\rangle = T\left\|+-+-+\right\rangle$
$\left\|+--++\right\rangle = T^2\left\|-+++-\right\rangle$	$\left\|-++-+\right\rangle = T^2\left\|+-+-+\right\rangle$
$\left\|++--+\right\rangle = T^3\left\|-+++-\right\rangle$	$\left\|+-++-\right\rangle = T^3\left\|+-+-+\right\rangle$
$\left\|+++--\right\rangle = T^4\left\|-+++-\right\rangle$	$\left\|-+-++\right\rangle = T^4\left\|+-+-+\right\rangle$

In other hand, the Hamiltonian and rotation are commutative. Easily we can reach to answers with solving Hamiltonian for only two states $\left|-+++-\right\rangle$ and $\left|+-+-+\right\rangle$:

$$H = J\begin{pmatrix} \Delta & 2(T^2+T^3) \\ 2(T^2+T^3) & 2(T^2+T^3)-3\Delta \end{pmatrix} = J\begin{pmatrix} \Delta & 0 \\ 0 & -3\Delta \end{pmatrix} + J\begin{pmatrix} 0 & 2(T^2+T^3) \\ 2(T^2+T^3) & 2(T^2+T^3) \end{pmatrix} = Jh$$

with the knowledge of that momentum is invariant of Hamiltonian, eigenvalus and eigenstates are:

$$p=0 \Rightarrow \begin{cases} \psi_1 = \dfrac{\sqrt{5}}{5}\sum_{i=1}^{5} T^i\left|-+++-\right\rangle \\ \psi_2 = \dfrac{\sqrt{5}}{5}\sum_{i=1}^{5} T^i\left|+-+-+\right\rangle \end{cases}$$

$$h\psi_1 = \frac{\sqrt{5}}{5}(\Delta\psi_1 + 4\psi_2)$$

$$h\psi_2 = \frac{\sqrt{5}}{5}(4\psi_1 + (4-3\Delta)\psi_2) \quad \Rightarrow h = \begin{pmatrix} \Delta & 4 \\ 4 & 4-3\Delta \end{pmatrix} \Rightarrow \lambda = J\left(-\Delta + 2 \pm 2\sqrt{\Delta^2 - 2\Delta + 5}\right)$$

$$\begin{cases} \lambda_1 = J\left(-\Delta + 2 - 2\sqrt{\Delta^2 - 2\Delta + 5}\right) \\ \lambda_2 = J\left(-\Delta + 2 + 2\sqrt{\Delta^2 - 2\Delta + 5}\right) \end{cases}$$

$$\Rightarrow \left|\Psi_1\right\rangle = \frac{2}{\sqrt{4 + (\Delta - 1 + \sqrt{\Delta^2 - 2\Delta + 5})^2}}(\psi_1 - \frac{\Delta - 1 + \sqrt{\Delta^2 - 2\Delta + 5}}{2}\psi_2)$$

$$\Rightarrow \left|\Psi_2\right\rangle = \frac{2}{\sqrt{4 + (\Delta - 1 - \sqrt{\Delta^2 - 2\Delta + 5})^2}}(\psi_1 - \frac{\Delta - 1 - \sqrt{\Delta^2 - 2\Delta + 5}}{2}\psi_2)$$

$$p=\frac{2\pi}{5}\Rightarrow\begin{cases}\psi_1=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{2\pi}{5}j}\left|-+++-\right\rangle\\ \psi_2=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{2\pi}{5}j}\left|+-+-+\right\rangle\end{cases}$$

$$h\psi_1=\frac{\sqrt{5}}{5}\left(\Delta\psi_1-4\cos\left(\frac{\pi}{5}\right)\psi_2\right)$$

$$h\psi_2=\frac{\sqrt{5}}{5}\left(-4\cos\left(\frac{\pi}{5}\right)\psi_1-\left(4\cos\left(\frac{\pi}{5}\right)+3\Delta\right)\psi_2\right)\Rightarrow$$

$$h=\begin{pmatrix}\Delta & -4\cos\left(\frac{\pi}{5}\right)\\ -4\cos\left(\frac{\pi}{5}\right) & -4\cos\left(\frac{\pi}{5}\right)-3\Delta\end{pmatrix}$$

$$\Rightarrow\lambda=J\left(-\Delta-2\cos\left(\frac{\pi}{5}\right)\pm2\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)$$

$$\begin{cases}\lambda_3=J\left(-\Delta-2\cos\left(\frac{\pi}{5}\right)-2\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)\\ \lambda_4=J\left(-\Delta-2\cos\left(\frac{\pi}{5}\right)+2\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)\end{cases}$$

$$\Rightarrow\left|\Psi_3\right\rangle=\frac{2\cos\left(\frac{\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+\left(\Delta+\cos\left(\frac{\pi}{5}\right)+\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)^2}}(\psi_1$$

$$+\frac{\Delta+\cos\left(\frac{\pi}{5}\right)+\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}}{2\cos\left(\frac{\pi}{5}\right)}\psi_2)$$

$$\Rightarrow\left|\Psi_4\right\rangle=\frac{2\cos\left(\frac{\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+\left(\Delta+\cos\left(\frac{\pi}{5}\right)-\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)^2}}(\psi_1$$

$$+\frac{\Delta+\cos\left(\frac{\pi}{5}\right)-\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}}{2\cos\left(\frac{\pi}{5}\right)}\psi_2)$$

$$p=\frac{4\pi}{5}\Rightarrow\begin{cases}\psi_1=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{4\pi}{5}j}\left|-+++-\right\rangle\\ \psi_2=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{4\pi}{5}j}\left|+-+-+\right\rangle\end{cases}$$

$$h\psi_1=\frac{\sqrt{5}}{5}\left(\Delta\psi_1+4\cos\left(\frac{2\pi}{5}\right)\psi_2\right)$$

$$h\psi_2=\frac{\sqrt{5}}{5}\left(4\cos\left(\frac{2\pi}{5}\right)\psi_1+\left(4\cos\left(\frac{2\pi}{5}\right)-3\Delta\right)\psi_2\right)\Rightarrow$$

$$h=\begin{pmatrix}\Delta & 4\cos\left(\frac{2\pi}{5}\right)\\ 4\cos\left(\frac{2\pi}{5}\right) & 4\cos\left(\frac{2\pi}{5}\right)-3\Delta\end{pmatrix}$$

$$\Rightarrow\lambda=J\left(-\Delta+2\cos\left(\frac{2\pi}{5}\right)\pm2\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)$$

$$\begin{cases}\lambda_5=J\left(-\Delta+2\cos\left(\frac{2\pi}{5}\right)-2\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)\\ \lambda_6=J\left(-\Delta-2\cos\left(\frac{2\pi}{5}\right)+2\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)\end{cases}$$

$$\Rightarrow\left|\Psi_5\right\rangle=\frac{2\cos\left(\frac{2\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{2\pi}{5}\right)+\left(\Delta-\cos\left(\frac{2\pi}{5}\right)-\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)^2}}(\psi_1$$
$$-\frac{\Delta-\cos\left(\frac{2\pi}{5}\right)-\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}}{2\cos\left(\frac{2\pi}{5}\right)}\psi_2)$$

$$\Rightarrow\left|\Psi_6\right\rangle=\frac{2\cos\left(\frac{2\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{2\pi}{5}\right)+\left(\Delta-\cos\left(\frac{2\pi}{5}\right)+\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)^2}}(\psi_1$$
$$-\frac{\Delta-\cos\left(\frac{2\pi}{5}\right)+\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}}{2\cos\left(\frac{2\pi}{5}\right)}\psi_2)$$

$$p=\frac{6\pi}{5}\Rightarrow\begin{cases}\psi_1=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{6\pi}{5}j}\left|-+++-\right\rangle=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{i\frac{4\pi}{5}j}\left|-+++-\right\rangle\\ \psi_2=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{6\pi}{5}j}\left|+-+-+\right\rangle=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{i\frac{4\pi}{5}j}\left|+-+-+\right\rangle\end{cases}$$

$$h\psi_1=\frac{\sqrt{5}}{5}\left(\Delta\psi_1+4\cos\left(\frac{2\pi}{5}\right)\psi_2\right)$$

$$h\psi_2=\frac{\sqrt{5}}{5}\left(4\cos\left(\frac{2\pi}{5}\right)\psi_1+\left(4\cos\left(\frac{2\pi}{5}\right)-3\Delta\right)\psi_2\right)\Rightarrow$$

$$h=\begin{pmatrix}\Delta & 4\cos\left(\frac{2\pi}{5}\right)\\ 4\cos\left(\frac{2\pi}{5}\right) & 4\cos\left(\frac{2\pi}{5}\right)-3\Delta\end{pmatrix}$$

$$\Rightarrow\lambda=J\left(-\Delta+2\cos\left(\frac{2\pi}{5}\right)\pm2\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)$$

$$\begin{cases}\lambda_7=J\left(-\Delta+2\cos\left(\frac{2\pi}{5}\right)-2\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)\\ \lambda_8=J\left(-\Delta-2\cos\left(\frac{2\pi}{5}\right)+2\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)\end{cases}$$

$$\Rightarrow\left|\Psi_7\right\rangle=\frac{2\cos\left(\frac{2\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{2\pi}{5}\right)+\left(\Delta-\cos\left(\frac{2\pi}{5}\right)-\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)^2}}\left(\psi_1-\frac{\Delta-\cos\left(\frac{2\pi}{5}\right)-\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}}{2\cos\left(\frac{2\pi}{5}\right)}\psi_2\right)$$

$$\Rightarrow\left|\Psi_8\right\rangle=\frac{2\cos\left(\frac{2\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{2\pi}{5}\right)+\left(\Delta-\cos\left(\frac{2\pi}{5}\right)+\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}\right)^2}}\left(\psi_1-\frac{\Delta-\cos\left(\frac{2\pi}{5}\right)+\sqrt{\Delta^2-2\Delta\cos\left(\frac{2\pi}{5}\right)+5\cos^2\left(\frac{2\pi}{5}\right)}}{2\cos\left(\frac{2\pi}{5}\right)}\psi_2\right)$$

$$p=\frac{8\pi}{5}\Rightarrow\begin{cases}\psi_1=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{8\pi}{5}j}\left|-+++-\right\rangle=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{i\frac{2\pi}{5}j}\left|-+++-\right\rangle\\ \psi_2=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{-i\frac{8\pi}{5}j}\left|+-+-+\right\rangle=\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^j e^{i\frac{2\pi}{5}j}\left|+-+-+\right\rangle\end{cases}$$

$$\begin{aligned}h\psi_1&=\frac{\sqrt{5}}{5}\left(\Delta\psi_1-4\cos\left(\frac{\pi}{5}\right)\psi_2\right)\\ h\psi_2&=\frac{\sqrt{5}}{5}\left(-4\cos\left(\frac{\pi}{5}\right)\psi_1-\left(4\cos\left(\frac{\pi}{5}\right)+3\Delta\right)\psi_2\right)\end{aligned}\Rightarrow$$

$$h=\begin{pmatrix}\Delta & -4\cos\left(\frac{\pi}{5}\right)\\ -4\cos\left(\frac{\pi}{5}\right) & -4\cos\left(\frac{\pi}{5}\right)-3\Delta\end{pmatrix}$$

$$\Rightarrow\lambda=J\left(-\Delta-2\cos\left(\frac{\pi}{5}\right)\pm 2\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)$$

$$\begin{cases}\lambda_9=J\left(-\Delta-2\cos\left(\frac{\pi}{5}\right)-2\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)\\ \lambda_{10}=J\left(-\Delta-2\cos\left(\frac{\pi}{5}\right)+2\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)\end{cases}$$

$$\Rightarrow\left|\Psi_9\right\rangle=\frac{2\cos\left(\frac{\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+\left(\Delta+\cos\left(\frac{\pi}{5}\right)+\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)^2}}(\psi_1+\frac{\Delta+\cos\left(\frac{\pi}{5}\right)+\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}}{2\cos\left(\frac{\pi}{5}\right)}\psi_2)$$

$$\Rightarrow\left|\Psi_{10}\right\rangle=\frac{2\cos\left(\frac{\pi}{5}\right)}{\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+\left(\Delta+\cos\left(\frac{\pi}{5}\right)-\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}\right)^2}}(\psi_1+\frac{\Delta+\cos\left(\frac{\pi}{5}\right)-\sqrt{\Delta^2+2\Delta\cos\left(\frac{\pi}{5}\right)+5\cos^2\left(\frac{\pi}{5}\right)}}{2\cos\left(\frac{\pi}{5}\right)}\psi_2)$$

The eigenvalues for these states in interval of 1 and -1 in order of lower to higher level are written:

$$\varepsilon_0 = J\left(-\Delta - 2\cos\left(\frac{\pi}{5}\right) - 2\sqrt{\Delta^2 + 2\Delta\cos\left(\frac{\pi}{5}\right) + 5\cos^2\left(\frac{\pi}{5}\right)} \right) \qquad : |\Psi_3\rangle, |\Psi_9\rangle$$

$$\varepsilon_1 = J\left(-\Delta + 2 - 2\sqrt{\Delta^2 - 2\Delta + 5} \right) \qquad : |\Psi_1\rangle$$

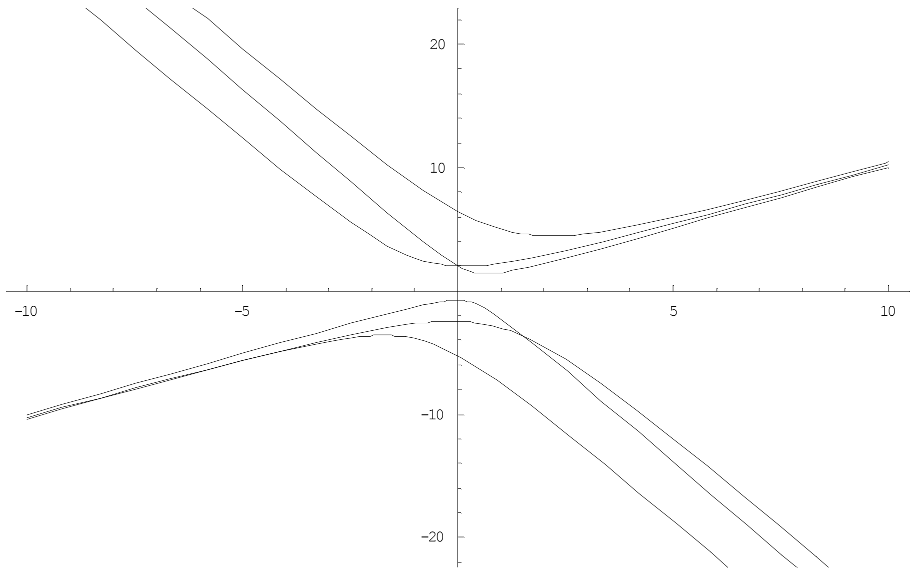
$$\varepsilon_2 = J\left(-\Delta + 2\cos\left(\frac{2\pi}{5}\right) - 2\sqrt{\Delta^2 - 2\Delta\cos\left(\frac{2\pi}{5}\right) + 5\cos^2\left(\frac{2\pi}{5}\right)} \right) \qquad : |\Psi_5\rangle, |\Psi_7\rangle$$

$$\varepsilon_3 = J\left(-\Delta - 2\cos\left(\frac{\pi}{5}\right) + 2\sqrt{\Delta^2 + 2\Delta\cos\left(\frac{\pi}{5}\right) + 5\cos^2\left(\frac{\pi}{5}\right)} \right) \qquad : |\Psi_4\rangle, |\Psi_{10}\rangle$$

$$\varepsilon_4 = J\left(-\Delta + 2\cos\left(\frac{2\pi}{5}\right) + 2\sqrt{\Delta^2 - 2\Delta\cos\left(\frac{2\pi}{5}\right) + 5\cos^2\left(\frac{2\pi}{5}\right)} \right) \qquad : |\Psi_6\rangle, |\Psi_8\rangle$$

$$\varepsilon_5 = J\left(-\Delta + 2 + 2\sqrt{\Delta^2 - 2\Delta + 5} \right) \qquad : |\Psi_2\rangle$$

The graph of this energy states in term of Δ is



2. Renormalization

Assume a chain with N particle. We can divide this chain to $\frac{N}{5}$ chain with 5 particle. So the Hamiltonian of system can be written with two part:

$$H = \sum_{I=1}^{\frac{N}{5}} H_I^0 + \sum_{I=1}^{\frac{N}{5}} H_{I,I+1}$$

where H_I^0 is internal Hamiltonian of each chain with 5 particle and it H_I^0 is what we calculate in last part. $H_{I,I+1}$ is interaction of I[th] chain with next chain. In this project, the system is considered in zero temperature, so the ground state is what we are looking for. As an approximation, we consider each sub-chain (I) a closed chain because we know the analytical solutions of this model.

The ground state of H_I^0, we have 4 degenerate state which two of them are $m=\frac{1}{2}$ and two of them have $m=-\frac{1}{2}$, which $m=-\frac{1}{2}$ states are same as $m=\frac{1}{2}$ states with reverse spins.

$$|1\rangle = A\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-i\frac{2\pi}{5}j}|-+++-\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-i\frac{2\pi}{5}j}|+-+-+\rangle\right) = A\left(\psi_1^{\frac{2\pi}{5},\frac{1}{2}} + B\psi_2^{\frac{2\pi}{5},\frac{1}{2}}\right)$$

$$|2\rangle = A\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{i\frac{2\pi}{5}j}|-+++-\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{i\frac{2\pi}{5}j}|+-+-+\rangle\right) = A\left(\psi_1^{-\frac{2\pi}{5},\frac{1}{2}} + B\psi_2^{-\frac{2\pi}{5},\frac{1}{2}}\right)$$

$$|3\rangle = A\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-i\frac{2\pi}{5}j}|+---+\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-i\frac{2\pi}{5}j}|-+-+-\rangle\right) = A\left(\psi_1^{\frac{2\pi}{5},-\frac{1}{2}} + B\psi_2^{\frac{2\pi}{5},-\frac{1}{2}}\right)$$

$$|4\rangle = A\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{i\frac{2\pi}{5}j}|+---+\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{i\frac{2\pi}{5}j}|--+-+\rangle\right) = A\left(\psi_1^{-\frac{2\pi}{5},-\frac{1}{2}} + B\psi_2^{-\frac{2\pi}{5},-\frac{1}{2}}\right)$$

In renormalization, the Hamiltonian of system will be written as effective Hamiltonian for image of system in eigenstates of state space of sub-chains and it will be done with P operator:

$$P = |1\rangle\langle\uparrow| + |2\rangle\langle\uparrow| + |3\rangle\langle\downarrow| + |4\rangle\langle\downarrow|$$

So effective Hamiltonian will be:

$$H_{eff} = P^t H P = \sum_{I=1}^{N/5} H_I^{0,eff} + \sum_{I=1}^{N/5} H_{I,I+1}^{eff}$$

So for each sub-chain effective Hamiltonian is

$$H_I^{0,eff} = P_I^t H_I^0 P_I = \left(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right|\right) H_I^0$$
$$\left(\left|1,I\right\rangle\left\langle\uparrow_I\right| + \left|2,I\right\rangle\left\langle\uparrow_I\right| + \left|3,I\right\rangle\left\langle\downarrow_I\right| + \left|4,I\right\rangle\left\langle\downarrow_I\right|\right)$$
$$= \left(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right|\right)$$
$$\left(H_I^0\left|1,I\right\rangle\left\langle\uparrow_I\right| + H_I^0\left|2,I\right\rangle\left\langle\uparrow_I\right| + H_I^0\left|3,I\right\rangle\left\langle\downarrow_I\right| + H_I^0\left|4,I\right\rangle\left\langle\downarrow_I\right|\right)$$
$$= \left(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right|\right)\varepsilon_0\left(\left|1,I\right\rangle\left\langle\uparrow_I\right| + \left|2,I\right\rangle\left\langle\uparrow_I\right| + \left|3,I\right\rangle\left\langle\downarrow_I\right| + \left|4,I\right\rangle\left\langle\downarrow_I\right|\right)$$
$$= 2\varepsilon_0\left(\left|\uparrow_I\right\rangle\left\langle\uparrow_I\right| + \left|\downarrow_I\right\rangle\left\langle\downarrow_I\right|\right) = 2\varepsilon_0 1_I$$

So effective Hamiltonian for interaction between sub-chains is

$$H_{I,I+1}^{eff} = P_{I+1}{}^t P_I^t H_{I,I+1} P_I P_{I+1}$$

For two neighbor sub-chain the interaction Hamiltonian is

$$H_{I,I+1} = J(\sigma_{5,I}^x \sigma_{1,I+1}^x + \sigma_{5,I}^y \sigma_{1,I+1}^y + \Delta\sigma_{5,I}^z \sigma_{1,I+1}^z)$$

and the effective interaction Hamiltonian will be:

$$H_{I,I+1}^{0,eff} = \left(\left|\uparrow_{I+1}\right\rangle\left\langle 1,I+1\right| + \left|\uparrow_{I+1}\right\rangle\left\langle 2,I+1\right| + \left|\downarrow_{I+1}\right\rangle\left\langle 3,I+1\right| + \left|\downarrow_{I+1}\right\rangle\left\langle 4,I+1\right|\right)$$
$$\left(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right|\right)\left[J\left(\sigma_{5,I}^x \sigma_{1,I+1}^x + \sigma_{5,I}^y \sigma_{1,I+1}^y + \Delta\sigma_{5,I}^z \sigma_{1,I+1}^z\right)\right]$$
$$\left(\left|1,I\right\rangle\left\langle\uparrow_I\right| + \left|2,I\right\rangle\left\langle\uparrow_I\right| + \left|3,I\right\rangle\left\langle\downarrow_I\right| + \left|4,I\right\rangle\left\langle\downarrow_I\right|\right)$$
$$\left(\left|1,I+1\right\rangle\left\langle\uparrow_{I+1}\right| + \left|2,I+1\right\rangle\left\langle\uparrow_{I+1}\right| + \left|3,I+1\right\rangle\left\langle\downarrow_{I+1}\right| + \left|4,I+1\right\rangle\left\langle\downarrow_{I+1}\right|\right)$$

Now we calculate each Pauli metrics:

$$\sigma_{5,I}^x\left|\tfrac{1}{2},p\right\rangle_I = \sigma_{5,I}^x\left[A\left(\psi_{1,I}^{p,\frac{1}{2}} + B\psi_{2,I}^{p,\frac{1}{2}}\right)\right]$$
$$= A\left[\sigma_5^x\left(\frac{\sqrt{5}}{5}\sum_{j=1}^5 T^j e^{-ipj}\left|-+++-\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^5 T^j e^{-ipj}\left|+-+-+\right\rangle\right)\right]_I$$
$$= A\left[\left(\frac{\sqrt{5}}{5}\sum_{j=1}^5 e^{-ipj}\sigma_5^x T^j\left|-+++-\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^5 e^{-ipj}\sigma_5^x T^j\left|+-+-+\right\rangle\right)\right]_I$$
$$= A\frac{\sqrt{5}}{5}\left[\left(\sum_{j=1}^5 e^{-ipj}\sigma_5^x T^j\left|-+++-\right\rangle + B\sum_{j=1}^5 e^{-ipj}\sigma_5^x T^j\left|+-+-+\right\rangle\right)\right]_I$$
$$= A\frac{\sqrt{5}}{5}\left[\sigma_5^x\left|-+++-\right\rangle + e^{-ip}\sigma_5^x\left|--+++\right\rangle + e^{-2ip}\sigma_5^x\left|+--++\right\rangle + e^{-3ip}\sigma_5^x\left|++--+\right\rangle\right.$$
$$+ e^{-4ip}\sigma_5^x\left|+++--\right\rangle + B\left(\sigma_5^x\left|+-+-+\right\rangle + e^{-ip}\sigma_5^x\left|++-+-\right\rangle + e^{-2ip}\sigma_5^x\left|-++-+\right\rangle\right.$$
$$\left.\left. + e^{-3ip}\sigma_5^x\left|+-++-\right\rangle + e^{-4ip}\sigma_5^x\left|-+-++\right\rangle\right)\right]_I$$

$$= A\frac{\sqrt{5}}{5}\Big[\big|-++++\big\rangle + e^{-ip}\big|--++-\big\rangle + e^{-2ip}\big|+--+-\big\rangle + e^{-3ip}\big|++---\big\rangle + e^{-4ip}\big|+++-+\big\rangle$$

$$+ B\Big(\big|+-+--\big\rangle + e^{-ip}\big|++-++\big\rangle + e^{-2ip}\big|-++--\big\rangle + e^{-3ip}\big|+-+++\big\rangle + e^{-4ip}\big|-+-+-\big\rangle\Big)\Big]_I$$

$$\sigma_{5,I}^{x}\big|-\tfrac{1}{2}, p\big\rangle_I = \sigma_{5,I}^{x}\Big[A\Big(\psi_{1,I}^{p,-\frac{1}{2}} + B\,\psi_{2,I}^{p,-\frac{1}{2}}\Big)\Big]$$

$$= A\frac{\sqrt{5}}{5}\Big[\big|+----\big\rangle + e^{-ip}\big|++--+\big\rangle + e^{-2ip}\big|-++-+\big\rangle + e^{-3ip}\big|--+++\big\rangle + e^{-4ip}\big|---+-\big\rangle$$

$$+ B\Big(\big|-+-++\big\rangle + e^{-ip}\big|--+--\big\rangle + e^{-2ip}\big|+--++\big\rangle + e^{-3ip}\big|-+---\big\rangle + e^{-4ip}\big|+-+-+\big\rangle\Big)\Big]$$

$$\sigma_{1,I+1}^{x}\big|\tfrac{1}{2}, p\big\rangle_{I+1} = \sigma_{1,I+1}^{x}\Big[A\Big(\psi_{1,I+1}^{p,\frac{1}{2}} + B\,\psi_{2,I+1}^{p,\frac{1}{2}}\Big)\Big]$$

$$= A\frac{\sqrt{5}}{5}\Big[\big|++++-\big\rangle + e^{-ip}\big|+-+++\big\rangle + e^{-2ip}\big|---++\big\rangle + e^{-3ip}\big|-+--+\big\rangle$$

$$+ e^{-4ip}\big|-++--\big\rangle + B\Big(\big|--+-+\big\rangle + e^{-ip}\big|-+-+-\big\rangle + e^{-2ip}\big|+++-+\big\rangle$$

$$+ e^{-3ip}\big|--++-\big\rangle + e^{-4ip}\big|++-++\big\rangle\Big)\Big]_{I+1}$$

$$\sigma_{1,I+1}^{x}\big|\tfrac{-1}{2}, p\big\rangle_{I+1} = \sigma_{1,I+1}^{x}\Big[A\Big(\psi_{1,I+1}^{p,-\frac{1}{2}} + B\,\psi_{2,I+1}^{p,-\frac{1}{2}}\Big)\Big]$$

$$= A\frac{\sqrt{5}}{5}\Big[\big|----+\big\rangle + e^{-ip}\big|-+---\big\rangle + e^{-2ip}\big|+++--\big\rangle + e^{-3ip}\big|+-++-\big\rangle$$

$$+ e^{-4ip}\big|+--++\big\rangle + B\Big(\big|++-+-\big\rangle + e^{-ip}\big|+-+-+\big\rangle + e^{-2ip}\big|---+-\big\rangle$$

$$+ e^{-3ip}\big|++--+\big\rangle + e^{-4ip}\big|--+--\big\rangle\Big)\Big]_{I+1}$$

$$P_I^{t}\sigma_{5,I}^{x}P_I$$

$$= \Big(\big|\uparrow_I\big\rangle\big\langle 1,I\big| + \big|\uparrow_I\big\rangle\big\langle 2,I\big| + \big|\downarrow_I\big\rangle\big\langle 3,I\big| + \big|\downarrow_I\big\rangle\big\langle 4,I\big|\Big)\sigma_{5,I}^{x}\Big(\big|1,I\big\rangle\big\langle\uparrow_I\big| + \big|2,I\big\rangle\big\langle\uparrow_I\big| + \big|3,I\big\rangle\big\langle\downarrow_I\big| + \big|4,I\big\rangle\big\langle\downarrow_I\big|\Big)$$

$$= \Big(\big|\uparrow_I\big\rangle\big\langle 1,I\big| + \big|\uparrow_I\big\rangle\big\langle 2,I\big| + \big|\downarrow_I\big\rangle\big\langle 3,I\big| + \big|\downarrow_I\big\rangle\big\langle 4,I\big|\Big)$$

$$\Big\{A\frac{\sqrt{5}}{5}\Big[\big|-++++\big\rangle + e^{-i\frac{2\pi}{5}}\big|--++-\big\rangle + e^{-2i\frac{2\pi}{5}}\big|+--+-\big\rangle + e^{-3i\frac{2\pi}{5}}\big|++---\big\rangle + e^{-4i\frac{2\pi}{5}}\big|+++-+\big\rangle$$

$$+ B\Big(\big|+-+--\big\rangle + e^{-i\frac{2\pi}{5}}\big|++-++\big\rangle + e^{-2i\frac{2\pi}{5}}\big|-++--\big\rangle + e^{-3i\frac{2\pi}{5}}\big|+-+++\big\rangle + e^{-4i\frac{2\pi}{5}}\big|-+-+-\big\rangle\Big)\Big]_I\big\langle\uparrow_I\big|$$

$$+ A\frac{\sqrt{5}}{5}\Big[\big|-++++\big\rangle + e^{i\frac{2\pi}{5}}\big|--++-\big\rangle + e^{2i\frac{2\pi}{5}}\big|+--+-\big\rangle + e^{3i\frac{2\pi}{5}}\big|++---\big\rangle + e^{4i\frac{2\pi}{5}}\big|+++-+\big\rangle$$

$$+ B\Big(\big|+-+--\big\rangle + e^{i\frac{2\pi}{5}}\big|++-++\big\rangle + e^{2i\frac{2\pi}{5}}\big|-++--\big\rangle + e^{3i\frac{2\pi}{5}}\big|+-+++\big\rangle + e^{4i\frac{2\pi}{5}}\big|-+-+-\big\rangle\Big)\Big]_I\big\langle\uparrow_I\big|$$

$$A\frac{\sqrt{5}}{5}\Big[\big|+----\big\rangle + e^{-i\frac{2\pi}{5}}\big|++--+\big\rangle + e^{-2i\frac{2\pi}{5}}\big|-++-+\big\rangle + e^{-3i\frac{2\pi}{5}}\big|--+++\big\rangle + e^{-4i\frac{2\pi}{5}}\big|---+-\big\rangle$$

$$+ B\Big(\big|-+-++\big\rangle + e^{-i\frac{2\pi}{5}}\big|--+--\big\rangle + e^{-2i\frac{2\pi}{5}}\big|+--++\big\rangle + e^{-3i\frac{2\pi}{5}}\big|-+---\big\rangle + e^{-4i\frac{2\pi}{5}}\big|+-+-+\big\rangle\Big)\Big]_I\big\langle\downarrow_I\big|$$

$$A\frac{\sqrt{5}}{5}\Big[\big|+----\big\rangle + e^{i\frac{2\pi}{5}}\big|++--+\big\rangle + e^{2i\frac{2\pi}{5}}\big|-++-+\big\rangle + e^{3i\frac{2\pi}{5}}\big|--+++\big\rangle + e^{4i\frac{2\pi}{5}}\big|---+-\big\rangle$$

$$+ B\Big(\big|-+-++\big\rangle + e^{i\frac{2\pi}{5}}\big|--+--\big\rangle + e^{2i\frac{2\pi}{5}}\big|+--++\big\rangle + e^{3i\frac{2\pi}{5}}\big|-+---\big\rangle + e^{4i\frac{2\pi}{5}}\big|+-+-+\big\rangle\Big)\Big]_I\big\langle\downarrow_I\big|\Big\}$$

$$
\begin{aligned}
&= \left(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right| \right) \\
&2A\frac{\sqrt{5}}{5}\Big\{ \Big[\left|-++++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--++-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|+--+-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|++---\right\rangle \\
&\qquad + \cos\left(\tfrac{2\pi}{5}\right)\left|+++-+\right\rangle + B\left(\left|+-+--\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|++-++\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|-++--\right\rangle \right. \\
&\qquad + \cos\left(\tfrac{4\pi}{5}\right)\left|+-+++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|-+-+-\right\rangle\Big]_I\left\langle\uparrow_I\right| \\
&\qquad + \Big[\left|+----\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|++--+\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|-++-+\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|--+++\right\rangle \\
&\qquad + \cos\left(\tfrac{2\pi}{5}\right)\left|---+-\right\rangle + B\left(\left|-+-++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--+--\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|+--++\right\rangle \right. \\
&\qquad \left. + \cos\left(\tfrac{4\pi}{5}\right)\left|-+---\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|+-+-+\right\rangle\right)\Big]_I\left\langle\downarrow_I\right| \Big\} \\
&= \frac{4A^2}{5}\Big\{ \left|\uparrow_I\right\rangle\Big[\left\langle -+++-\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle --+++\right| + \cos\left(\tfrac{4\pi}{5}\right)\left\langle +--++\right| + \cos\left(\tfrac{4\pi}{5}\right)\left\langle ++--+\right| \\
&\qquad + \cos\left(\tfrac{2\pi}{5}\right)\left\langle +++--\right| + B\left(\left\langle +-+-+\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle ++-+-\right| + \cos\left(\tfrac{4\pi}{5}\right)\left\langle -++-+\right| \right. \\
&\qquad \left. + \cos\left(\tfrac{4\pi}{5}\right)\left\langle +-++-\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle -+-++\right|\right) \\
&\qquad + \left|\downarrow_I\right\rangle\Big[\left\langle +---+\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle ++---\right| + \cos\left(\tfrac{4\pi}{5}\right)\left\langle -++--\right| + \cos\left(\tfrac{4\pi}{5}\right)\left\langle --++-\right| \\
&\qquad + \cos\left(\tfrac{2\pi}{5}\right)\left\langle ---++\right| + B\left(\left\langle -+-+-\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle --+-+\right| + \cos\left(\tfrac{4\pi}{5}\right)\left\langle +--+-\right| \right. \\
&\qquad \left. + \cos\left(\tfrac{4\pi}{5}\right)\left\langle -+--+\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle +-+--\right|\right)\Big]_I \Big\} \\
&\Big\{ \Big[\left|-++++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--++-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|+--+-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|++---\right\rangle \\
&\qquad + \cos\left(\tfrac{2\pi}{5}\right)\left|+++-+\right\rangle + B\left(\left|+-+--\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|++-++\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|-++--\right\rangle \right. \\
&\qquad + \cos\left(\tfrac{4\pi}{5}\right)\left|+-+++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|-+-+-\right\rangle\Big]_I\left\langle\uparrow_I\right| \\
&\qquad + \Big[\left|+----\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|++--+\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|-++-+\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|--+++\right\rangle \\
&\qquad + \cos\left(\tfrac{2\pi}{5}\right)\left|---+-\right\rangle + B\left(\left|-+-++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--+--\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|+--++\right\rangle \right. \\
&\qquad \left. + \cos\left(\tfrac{4\pi}{5}\right)\left|-+---\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|+-+-+\right\rangle\right)\Big]_I\left\langle\downarrow_I\right| \Big\} \\
&\qquad = \frac{8A^2}{5}\left(\cos\left(\tfrac{2\pi}{5}\right)\cos\left(\tfrac{4\pi}{5}\right) + B\cos^2\left(\tfrac{4\pi}{5}\right) + B^2\cos\left(\tfrac{2\pi}{5}\right)\right)\left(\left|\downarrow_I\right\rangle\left\langle\uparrow_I\right| + \left|\uparrow_I\right\rangle\left\langle\downarrow_I\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&P^t_{I+1}\sigma^x_{1,I+1}P_{I+1} \\
&= \left(\left|\uparrow_{I+1}\right\rangle\left\langle 1,I+1\right| + \left|\uparrow_{I+1}\right\rangle\left\langle 2,I+1\right| + \left|\downarrow_{I+1}\right\rangle\left\langle 3,I+1\right| + \left|\downarrow_{I+1}\right\rangle\left\langle 4,I+1\right| \right) \\
&\qquad\qquad \sigma^x_{1,I+1}\left(\left|1,I+1\right\rangle\left\langle\uparrow_{I+1}\right| + \left|2,I+1\right\rangle\left\langle\uparrow_{I+1}\right| + \left|3,I+1\right\rangle\left\langle\downarrow_{I+1}\right| + \left|4,I+1\right\rangle\left\langle\downarrow_{I+1}\right| \right)
\end{aligned}
$$

$$= \left(\left|\uparrow_{I+1}\right\rangle\langle 1, I+1| + \left|\uparrow_{I+1}\right\rangle\langle 2, I+1| + \left|\downarrow_{I+1}\right\rangle\langle 3, I+1| + \left|\downarrow_{I+1}\right\rangle\langle 4, I+1|\right)$$

$$\Big\{ 2A\frac{\sqrt{5}}{5}\Big[\big[|++++-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|-++++\rangle + \cos\left(\tfrac{4\pi}{5}\right)|---++\rangle + \cos\left(\tfrac{4\pi}{5}\right)|-+--+\rangle$$

$$+ \cos\left(\tfrac{2\pi}{5}\right)|-++--\rangle + B\big(|--+-+\rangle + \cos\left(\tfrac{2\pi}{5}\right)|-+-+-\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+++-+\rangle$$

$$+ \cos\left(\tfrac{4\pi}{5}\right)|--++-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|++-++\rangle\big)\Big]_{I+1}\left\langle\uparrow_{I+1}\right|$$

$$+ 2A\frac{\sqrt{5}}{5}\Big[\big(|----+\rangle + \cos\left(\tfrac{2\pi}{5}\right)|-+---\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+++--\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+-++-\rangle$$

$$+ \cos\left(\tfrac{2\pi}{5}\right)|+--++\rangle + B\big(|++-+-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|+-+-+\rangle + \cos\left(\tfrac{4\pi}{5}\right)|---+-\rangle$$

$$+ \cos\left(\tfrac{4\pi}{5}\right)|++--+\rangle + \cos\left(\tfrac{2\pi}{5}\right)|--+--\rangle\big)\Big]_{I+1}\left\langle\downarrow_{I+1}\right|\Big\}$$

$$= \frac{4A^2}{5}\Big\{ \left|\uparrow_{I+1}\right\rangle\Big[\big(\langle-++++-| + \cos\left(\tfrac{2\pi}{5}\right)\langle--+++| + \cos\left(\tfrac{4\pi}{5}\right)\langle+--++| + \cos\left(\tfrac{4\pi}{5}\right)\langle+++--+|$$

$$+ \cos\left(\tfrac{2\pi}{5}\right)\langle++++--| + B\ \big(\langle+-+-+| + \cos\left(\tfrac{2\pi}{5}\right)\langle++-+-| + \cos\left(\tfrac{4\pi}{5}\right)\langle-++-+|$$

$$+ \cos\left(\tfrac{4\pi}{5}\right)\langle+-++-| + \cos\left(\tfrac{2\pi}{5}\right)\langle-+-++|\big)\Big]_{I+1}$$

$$+ \left|\downarrow_{I+1}\right\rangle\Big[\langle+---+| + \cos\left(\tfrac{2\pi}{5}\right)\langle++---| + \cos\left(\tfrac{4\pi}{5}\right)\langle-++--| + \cos\left(\tfrac{4\pi}{5}\right)\langle--++-|$$

$$+ \cos\left(\tfrac{2\pi}{5}\right)\langle---++| + B\ \big(\langle-+-+-| + \cos\left(\tfrac{2\pi}{5}\right)\langle--+-+| + \cos\left(\tfrac{4\pi}{5}\right)\langle+--+-|$$

$$+ \cos\left(\tfrac{4\pi}{5}\right)\langle-+--+| + \cos\left(\tfrac{2\pi}{5}\right)\langle+-+--|\big)\Big]_{I+1}\Big\}$$

$$\Big\{ \Big[\big[|++++-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|-++++\rangle + \cos\left(\tfrac{4\pi}{5}\right)|---++\rangle + \cos\left(\tfrac{4\pi}{5}\right)|-+--+\rangle$$

$$+ \cos\left(\tfrac{2\pi}{5}\right)|-++--\rangle + B\big(|--+-+\rangle + \cos\left(\tfrac{2\pi}{5}\right)|-+-+-\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+++-+\rangle$$

$$+ \cos\left(\tfrac{4\pi}{5}\right)|--++-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|++-++\rangle\big)\Big]_{I+1}\left\langle\uparrow_{I+1}\right|$$

$$+ \Big[|----+\rangle + \cos\left(\tfrac{2\pi}{5}\right)|-+---\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+++--\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+-++-\rangle$$

$$+ \cos\left(\tfrac{2\pi}{5}\right)|+--++\rangle + B\big(|++-+-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|+-+-+\rangle + \cos\left(\tfrac{4\pi}{5}\right)|---+-\rangle$$

$$+ \cos\left(\tfrac{4\pi}{5}\right)|++--+\rangle + \cos\left(\tfrac{2\pi}{5}\right)|--+--\rangle\big)\Big]_{I+1}\left\langle\downarrow_{I+1}\right|\Big\}$$

$$= \frac{8A^2}{5}\left(\cos\left(\tfrac{2\pi}{5}\right)\cos\left(\tfrac{4\pi}{5}\right) + B\cos^2\left(\tfrac{4\pi}{5}\right) + B^2\cos\left(\tfrac{2\pi}{5}\right)\right)\left(\left|\downarrow_{I+1}\right\rangle\left\langle\uparrow_{I+1}\right| + \left|\uparrow_{I+1}\right\rangle\left\langle\downarrow_{I+1}\right|\right)$$

$$\sigma^y_{5,I}\left|\tfrac{1}{2}, p\right\rangle_I = \sigma^y_{5,I}\left[A\left(\psi^{p,\frac{1}{2}}_{1,I} + B\psi^{p,\frac{1}{2}}_{2,I}\right)\right]$$

$$= A\left[\sigma^y_{5,I}\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}|-+++-\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}|+-+-+\rangle\right)\right]_I$$

$$= A\left[\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} e^{-ipj}\sigma^y_5 T^j|-+++-\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} e^{-ipj}\sigma^y_5 T^j|+-+-+\rangle\right)\right]_I$$

$$= A\frac{\sqrt{5}}{5}\left[\left(\sum_{j=1}^{5} e^{-ipj}\sigma^y_5 T^j|-+++-\rangle + B\sum_{j=1}^{5} e^{-ipj}\sigma^y_5 T^j|+-+-+\rangle\right)\right]_I$$

$$= Ai\frac{\sqrt{5}}{5}\Big[-\left|-++++\right\rangle + e^{-ip}\left|---+-\right\rangle + e^{-2ip}\left|+--+-\right\rangle + e^{-3ip}\left|++---\right\rangle - e^{-4ip}\left|+++-+\right\rangle$$
$$+ B\Big(\left|+-+--\right\rangle - e^{-ip}\left|++-++\right\rangle + e^{-2ip}\left|-++--\right\rangle - e^{-3ip}\left|+-+++\right\rangle + e^{-4ip}\left|-+-+-\right\rangle\Big)\Big]_I$$

$$\sigma^y_{5,I}\left|\tfrac{-1}{2}, p\right\rangle_I = \sigma^y_{5,I}\Big[A\Big(\psi^{p,-\frac{1}{2}}_{1,I} + B\psi^{p,-\frac{1}{2}}_{2,I}\Big)\Big]$$
$$= A\left[\sigma^y_{5,I}\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}\left|+---+\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}\left|-+-+-\right\rangle\right)\right]_I$$
$$= Ai\frac{\sqrt{5}}{5}\Big[\Big(\left|+----\right\rangle - e^{-ip}\left|++--+\right\rangle - e^{-2ip}\left|-++-+\right\rangle - e^{-3ip}\left|--+++\right\rangle + e^{-4ip}\left|---+-\right\rangle$$
$$+ B\Big(-\left|-+-++\right\rangle + e^{-ip}\left|--+--\right\rangle - e^{-2ip}\left|+--++\right\rangle + e^{-3ip}\left|-+---\right\rangle - e^{-4ip}\left|+-+-+\right\rangle\Big)\Big]_I$$

$$\sigma^y_{1,I+1}\left|\tfrac{1}{2}, p\right\rangle_{I+1} = \sigma^y_{1,I+1}\Big[A\Big(\psi^{p,\frac{1}{2}}_{1,I+1} + B\psi^{p,\frac{1}{2}}_{2,I+1}\Big)\Big]$$
$$= A\left[\sigma^y_{1,I+1}\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}\left|-+++-\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}\left|+-+-+\right\rangle\right)\right]_{I+1}$$
$$= A\left[\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} e^{-ipj}\sigma^y_1 T^j\left|-+++-\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} e^{-ipj}\sigma^y_1 T^j\left|+-+-+\right\rangle\right)\right]_{I+1}$$
$$= Ai\frac{\sqrt{5}}{5}\Big[-\left|++++-\right\rangle - e^{-ip}\left|+-+++\right\rangle + e^{-2ip}\left|---++\right\rangle + e^{-3ip}\left|-+--+\right\rangle + e^{-4ip}\left|-++--\right\rangle$$
$$+ B\Big(\left|--+-+\right\rangle + e^{-ip}\left|-+-+-\right\rangle - e^{-2ip}\left|+++-+\right\rangle + e^{-3ip}\left|--++-\right\rangle - e^{-4ip}\left|++-++\right\rangle\Big)\Big]_{I+1}$$

$$\sigma^y_{1,I+1}\left|\tfrac{-1}{2}, p\right\rangle_{I+1} = \sigma^y_{1,I+1}\Big[A\Big(\psi^{p,\frac{1}{2}}_{1,I+1} + B\psi^{p,\frac{1}{2}}_{2,I+1}\Big)\Big]$$
$$= A\left[\sigma^y_{1,I+1}\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}\left|+---+\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} T^j e^{-ipj}\left|-+-+-\right\rangle\right)\right]_{I+1}$$
$$= A\left[\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5} e^{-ipj}\sigma^y_1 T^j\left|+---+\right\rangle + B\frac{\sqrt{5}}{5}\sum_{j=1}^{5} e^{-ipj}\sigma^y_1 T^j\left|-+-+-\right\rangle\right)\right]_{I+1}$$
$$= Ai\frac{\sqrt{5}}{5}\Big[\Big(\left|----+\right\rangle + e^{-ip}\left|-+---\right\rangle - e^{-2ip}\left|+++--\right\rangle - e^{-3ip}\left|+-++-\right\rangle - e^{-4ip}\left|+--++\right\rangle$$
$$+ B\Big(-\left|++-+-\right\rangle - e^{-ip}\left|+-+-+\right\rangle + e^{-2ip}\left|---+-\right\rangle - e^{-3ip}\left|+---+\right\rangle + e^{-4ip}\left|--+--\right\rangle\Big)\Big]_{I+1}$$

$$P^t_I\sigma^{5,I}_y P_I$$
$$= \Big(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right|\Big)\sigma^y_{5,I}\Big(\left|1,I\right\rangle\left\langle\uparrow_I\right| + \left|2,I\right\rangle\left\langle\uparrow_I\right| + \left|3,I\right\rangle\left\langle\downarrow_I\right| + \left|4,I\right\rangle\left\langle\downarrow_I\right|\Big)$$

$$= \left(\left|\uparrow_I\right\rangle\left\langle 1,I\right| + \left|\uparrow_I\right\rangle\left\langle 2,I\right| + \left|\downarrow_I\right\rangle\left\langle 3,I\right| + \left|\downarrow_I\right\rangle\left\langle 4,I\right|\right)$$

$$\left\{2Ai\frac{\sqrt{5}}{5}\left[-\left|-++++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--++-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|+--+-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|++---\right\rangle\right.\right.$$
$$- \cos\left(\tfrac{2\pi}{5}\right)\left|+++-+\right\rangle + B\left(\left|+-+--\right\rangle - \cos\left(\tfrac{2\pi}{5}\right)\left|++-++\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|-++--\right\rangle\right.$$
$$\left.\left.- \cos\left(\tfrac{4\pi}{5}\right)\left|+-+++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|-+-+-\right\rangle\right)\right]\left\langle\uparrow_I\right|$$

$$+ 2Ai\frac{\sqrt{5}}{5}\left[\left|+----\right\rangle - \cos\left(\tfrac{2\pi}{5}\right)\left|++--+\right\rangle - \cos\left(\tfrac{4\pi}{5}\right)\left|-++-+\right\rangle - \cos\left(\tfrac{4\pi}{5}\right)\left|--+++\right\rangle\right.$$
$$+ \cos\left(\tfrac{2\pi}{5}\right)\left|---+-\right\rangle + B\left(-\left|-+-++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--+--\right\rangle - \cos\left(\tfrac{4\pi}{5}\right)\left|+--++\right\rangle\right.$$
$$\left.\left.+ \cos\left(\tfrac{4\pi}{5}\right)\left|-+---\right\rangle - \cos\left(\tfrac{2\pi}{5}\right)\left|+-+-+\right\rangle\right]_I\left\langle\downarrow_I\right|$$

$$= i\frac{4A^2}{5}\left\{\ \left|\uparrow_I\right\rangle\left[\left(\left\langle -+++-\right| + \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle --+++\right| + \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle +--++\right| + \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle ++--+\right|\right.\right.\right.$$
$$+ \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle +++--\right| + B\ \left(\left\langle +-+-+\right| + \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle ++-+-\right| + \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle -++-+\right|\right.$$
$$\left.\left.+ \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle +-++-\right| + \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle -+-++\right|\right)\right]_I$$
$$+ \left|\downarrow_{I+1}\right\rangle\left[\left\langle +---+\right| + \cos\left(\tfrac{2\pi}{5}\right)\left\langle +++--\right| + \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle -++--\right| + \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle --++-\right|\right.$$
$$+ \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle ---++\right| + B\ \left(\left\langle -+-+-\right| + \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle --+-+\right| + \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle +--+-\right|\right.$$
$$\left.\left.\left.+ \cos\left(\tfrac{4\pi}{5}\right)\ \left\langle -+--+\right| + \cos\left(\tfrac{2\pi}{5}\right)\ \left\langle +-+--\right|\right)\right]_I\ \right\}$$

$$\left\{\ \left[-\left|-++++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--++-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|+--+-\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|++---\right\rangle\right.\right.$$
$$- \cos\left(\tfrac{2\pi}{5}\right)\left|+++-+\right\rangle + B\left(\left|+-+--\right\rangle - \cos\left(\tfrac{2\pi}{5}\right)\left|++-++\right\rangle + \cos\left(\tfrac{4\pi}{5}\right)\left|-++--\right\rangle\right.$$
$$\left.\left.- \cos\left(\tfrac{4\pi}{5}\right)\left|+-+++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|-+-+-\right\rangle\right)\right]\left\langle\uparrow_I\right|$$
$$+ \left[\left|+----\right\rangle - \cos\left(\tfrac{2\pi}{5}\right)\left|++--+\right\rangle - \cos\left(\tfrac{4\pi}{5}\right)\left|-++-+\right\rangle - \cos\left(\tfrac{4\pi}{5}\right)\left|--+++\right\rangle\right.$$
$$+ \cos\left(\tfrac{2\pi}{5}\right)\left|---+-\right\rangle + B\left(-\left|-+-++\right\rangle + \cos\left(\tfrac{2\pi}{5}\right)\left|--+--\right\rangle - \cos\left(\tfrac{4\pi}{5}\right)\left|+--++\right\rangle\right.$$
$$\left.\left.\left.+ \cos\left(\tfrac{4\pi}{5}\right)\left|-+---\right\rangle - \cos\left(\tfrac{2\pi}{5}\right)\left|+-+-+\right\rangle\right]_I\ \right\}\left\langle\downarrow_I\right|$$
$$= i\frac{8A^2}{5}\left[\left(\cos\left(\tfrac{2\pi}{5}\right)\cos\left(\tfrac{4\pi}{5}\right) + B\cos^2\left(\tfrac{4\pi}{5}\right) + B^2\cos\left(\tfrac{2\pi}{5}\right)\right)\left(\left|\downarrow_I\right\rangle\left\langle\uparrow_I\right| - \left|\uparrow_I\right\rangle\left\langle\downarrow_I\right|\right)\right]$$

$$P_{I+1}^t\sigma_y^{1,I+1}P_{I+1}$$
$$= \left(\left|\uparrow_{I+1}\right\rangle\left\langle 1,I+1\right| + \left|\uparrow_{I+1}\right\rangle\left\langle 2,I+1\right| + \left|\downarrow_{I+1}\right\rangle\left\langle 3,I+1\right| + \left|\downarrow_{I+1}\right\rangle\left\langle 4,I+1\right|\right)\sigma_{1,I+1}^y$$
$$\left(\left|1,I+1\right\rangle\left\langle\uparrow_{I+1}\right| + \left|2,I+1\right\rangle\left\langle\uparrow_{I+1}\right| + \left|3,I+1\right\rangle\left\langle\downarrow_{I+1}\right| + \left|4,I+1\right\rangle\left\langle\downarrow_{I+1}\right|\right)$$
$$= i\frac{8A^2}{5}\left[\left(\cos\left(\tfrac{2\pi}{5}\right)\cos\left(\tfrac{4\pi}{5}\right) + B\cos^2\left(\tfrac{4\pi}{5}\right) + B^2\cos\left(\tfrac{2\pi}{5}\right)\right)\left(\left|\downarrow_{I+1}\right\rangle\left\langle\uparrow_{I+1}\right| - \left|\uparrow_{I+1}\right\rangle\left\langle\downarrow_{I+1}\right|\right)\right]$$

$$\sigma^z_{5,I}\left|\tfrac{1}{2},p\right\rangle_I=\sigma^z_{5,I}\left[A\left(\psi^{p,\frac{1}{2}}_{1,I}+B\psi^{p,\frac{1}{2}}_{2,I}\right)\right]$$

$$=A\left[\sigma^z_5\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|-+++-\right\rangle+B\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|+-+-+\right\rangle\right)\right]_I$$

$$=A\left[\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5}e^{-ipj}\sigma^z_5T^j\left|-+++-\right\rangle+B\frac{\sqrt{5}}{5}\sum_{j=1}^{5}e^{-ipj}\sigma^z_5T^j\left|+-+-+\right\rangle\right)\right]_I$$

$$=A\frac{\sqrt{5}}{5}\left[\left(\sum_{j=1}^{5}e^{-ipj}\sigma^z_5T^j\left|-+++-\right\rangle+B\sum_{j=1}^{5}e^{-ipj}\sigma^z_5T^j\left|+-+-+\right\rangle\right)\right]_I$$

$$=A\frac{\sqrt{5}}{5}\Big[-\left|-+++-\right\rangle+e^{-ip}\left|--+++\right\rangle+e^{-2ip}\left|+--++\right\rangle+e^{-3ip}\left|++--+\right\rangle-e^{-4ip}\left|+++--\right\rangle$$
$$+B\left(\left|+-+-+\right\rangle-e^{-ip}\left|++-+-\right\rangle+e^{-2ip}\left|-++-+\right\rangle-e^{-3ip}\left|+-++-\right\rangle+e^{-4ip}\left|-+-++\right\rangle\right)\Big]_I$$

$$\sigma^z_{5,I}\left|\tfrac{-1}{2},p\right\rangle_I=\sigma^z_{5,I}\left[A\left(\psi^{p,\frac{-1}{2}}_{1,I}+B\psi^{p,\frac{-1}{2}}_{2,I}\right)\right]$$

$$=A\left[\sigma^z_5\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|+---+\right\rangle+B\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|-+-+-\right\rangle\right)\right]_I$$

$$=A\frac{\sqrt{5}}{5}\Big[\left|+---+\right\rangle-e^{-ip}\left|++---\right\rangle-e^{-2ip}\left|-++--\right\rangle-e^{-3ip}\left|--++-\right\rangle+e^{-4ip}\left|---++\right\rangle$$
$$+B\left(-\left|-+-+-\right\rangle+e^{-ip}\left|--+-+\right\rangle-e^{-2ip}\left|+--+-\right\rangle+e^{-3ip}\left|-+--+\right\rangle-e^{-4ip}\left|+-+--\right\rangle\right)\Big]_I$$

$$\sigma^z_{1,I+1}\left|\tfrac{1}{2},p\right\rangle_{I+1}=\sigma^z_{1,I+1}\left[A\left(\psi^{p,\frac{1}{2}}_{1,I+1}+B\psi^{p,\frac{1}{2}}_{2,I+1}\right)\right]$$

$$=A\left[\sigma^z_1\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|-+++-\right\rangle+B\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|+-+-+\right\rangle\right)\right]_{I+1}$$

$$=A\left[\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5}e^{-ipj}\sigma^z_1T^j\left|-+++-\right\rangle+B\frac{\sqrt{5}}{5}\sum_{j=1}^{5}e^{-ipj}\sigma^z_1T^j\left|+-+-+\right\rangle\right)\right]_{I+1}$$

$$=A\frac{\sqrt{5}}{5}\left[\left(\sum_{j=1}^{5}e^{-ipj}\sigma^z_1T^j\left|-+++-\right\rangle+B\sum_{j=1}^{5}e^{-ipj}\sigma^z_1T^j\left|+-+-+\right\rangle\right)\right]_{I+1}$$

$$=A\frac{\sqrt{5}}{5}\Big[-\left|-+++-\right\rangle-e^{-ip}\left|--+++\right\rangle+e^{-2ip}\left|+--++\right\rangle+e^{-3ip}\left|++--+\right\rangle+e^{-4ip}\left|+++--\right\rangle$$
$$+B\left(\left|+-+-+\right\rangle+e^{-ip}\left|++-+-\right\rangle-e^{-2ip}\left|-++-+\right\rangle+e^{-3ip}\left|+-++-\right\rangle-e^{-4ip}\left|-+-++\right\rangle\right)\Big]_{I+1}$$

$$\sigma^z_{1,I+1}\left|\tfrac{-1}{2},p\right\rangle_{I+1}=\sigma^z_{1,I+1}\left[A\left(\psi^{p,\frac{-1}{2}}_{1,I+1}+B\psi^{p,\frac{-1}{2}}_{2,I+1}\right)\right]$$

$$=A\left[\sigma^z_1\left(\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|+---+\right\rangle+B\frac{\sqrt{5}}{5}\sum_{j=1}^{5}T^je^{-ipj}\left|-+-+-\right\rangle\right)\right]_{I+1}$$

$$=A\frac{\sqrt{5}}{5}\Big[\left|+---+\right\rangle+e^{-ip}\left|++---\right\rangle-e^{-2ip}\left|-++--\right\rangle-e^{-3ip}\left|--++-\right\rangle-e^{-4ip}\left|---++\right\rangle$$
$$+B\left(-\left|-+-+-\right\rangle-e^{-ip}\left|--+-+\right\rangle+e^{-2ip}\left|+--+-\right\rangle-e^{-3ip}\left|-+--+\right\rangle+e^{-4ip}\left|+-+--\right\rangle\right)\Big]_{I+1}$$

$P^t_I\sigma^{5,I}_zP_I$

$$
= \left(|\uparrow_I\rangle\langle 1, I| + |\uparrow_I\rangle\langle 2, I| + |\downarrow_I\rangle\langle 3, I| + |\downarrow_I\rangle\langle 4, I| \right)
$$

$$
\Big\{ 2A\frac{\sqrt{5}}{5}\Big[-|-+++-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|--+++\rangle + \cos\left(\tfrac{4\pi}{5}\right)|+--++\rangle + \cos\left(\tfrac{4\pi}{5}\right)|++--+\rangle
$$

$$
- \cos\left(\tfrac{2\pi}{5}\right)|+++--\rangle
$$

$$
+ B\Big(|+-+-+\rangle - \cos\left(\tfrac{2\pi}{5}\right)|++-+-\rangle + \cos\left(\tfrac{4\pi}{5}\right)|-++-+\rangle - \cos\left(\tfrac{4\pi}{5}\right)|+-++-\rangle
$$

$$
+ \cos\left(\tfrac{2\pi}{5}\right)|-+-++\rangle \Big) \Big] \langle\uparrow_I|
$$

$$
+ 2A\frac{\sqrt{5}}{5}\Big[|+---+\rangle - \cos\left(\tfrac{2\pi}{5}\right)|++---\rangle - \cos\left(\tfrac{4\pi}{5}\right)|-++--\rangle - \cos\left(\tfrac{4\pi}{5}\right)|--++-\rangle
$$

$$
+ \cos\left(\tfrac{2\pi}{5}\right)|---++\rangle + B\Big(-|-+-+-\rangle + \cos\left(\tfrac{2\pi}{5}\right)|--+-+\rangle - \cos\left(\tfrac{4\pi}{5}\right)|+--+-\rangle
$$

$$
+ \cos\left(\tfrac{4\pi}{5}\right)|-+--+\rangle - \cos\left(\tfrac{2\pi}{5}\right)|+-+--\rangle \Big] \langle\downarrow_I|
$$

$$
= \frac{4A^2}{5}\left[2\cos^2\left(\tfrac{4\pi}{5}\right) - 1 + B^2 \right]\left(|\uparrow_I\rangle\langle\uparrow_I| - |\downarrow_I\rangle\langle\downarrow_I| \right)
$$

$$
H^{0,eff}_{I,I+1} = J\left(\frac{8A^2}{5}\right)^2 \left(\cos\left(\tfrac{2\pi}{5}\right)\cos\left(\tfrac{4\pi}{5}\right) + B\cos^2\left(\tfrac{4\pi}{5}\right) + B^2\cos\left(\tfrac{2\pi}{5}\right) \right)^2
$$

$$
\Big[\left(|\uparrow_I\rangle\langle\downarrow_I| + |\downarrow_I\rangle\langle\uparrow_I| \right)\left(|\uparrow_{I+1}\rangle\langle\downarrow_{I+1}| + |\downarrow_{I+1}\rangle\langle\uparrow_{I+1}| \right) + i\left(|\uparrow_I\rangle\langle\downarrow_I| - |\downarrow_I\rangle\langle\uparrow_I| \right) i\left(|\uparrow_{I+1}\rangle\langle\downarrow_{I+1}| - |\downarrow_{I+1}\rangle\langle\uparrow_{I+1}| \right) \Big]
$$

$$
+ \Delta\left(\frac{4A^2}{5}\right)^2 \left[2\cos^2\left(\tfrac{4\pi}{5}\right) - 1 + B^2 \right]^2 \left(|\uparrow_I\rangle\langle\uparrow_I| - |\downarrow_I\rangle\langle\downarrow_I| \right)\left(|\uparrow_{I+1}\rangle\langle\uparrow_{I+1}| - |\downarrow_{I+1}\rangle\langle\downarrow_{I+1}| \right)
$$

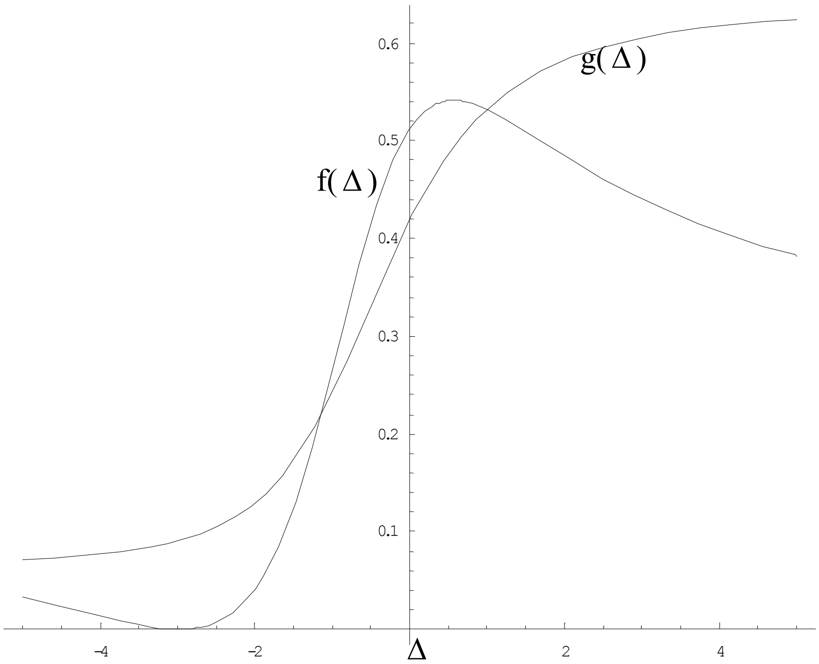
$$
= f(\Delta)J\left[\sigma_x^I\sigma_x^{I+1} + \sigma_y^I\sigma_y^{I+1} \right] + \Delta g(\Delta)J\sigma_z^I\sigma_z^{I+1}
$$

where

$$
f(\Delta) = \left(\frac{8A^2}{5}\right)^2 \left(\cos\left(\tfrac{2\pi}{5}\right)\cos\left(\tfrac{4\pi}{5}\right) + B\cos^2\left(\tfrac{4\pi}{5}\right) + B^2\cos\left(\tfrac{2\pi}{5}\right) \right)^2
$$

$$
g(\Delta) = \left(\frac{4A^2}{5}\right)^2 \left[2\cos^2\left(\tfrac{4\pi}{5}\right) - 1 + B^2 \right]^2
$$

Fixed points are where your system will decrease to finite chain system where the effective Hamiltonian is similar to finite chain Hamiltonian. In the next graph, f and g are scratch in term of Δ and the value for fixed points are driven:



The Δ values for fix points are

Δ=1.0000

Δ=-1.1545

Δ=0

which prove our expectation that the fix points are in the region of symmetry and this result shows this calculates are correct.